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THE UNIT BALL OF $\mathcal{L}({}^{2}\mathbb{R}^{2}_{h(w)})$

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ABSTRACT. We classify the extreme bilinear forms of the unit ball of the space of bilinear forms on \mathbb{R}^2 with hexagonal norms.

1. Introduction

We write B_E for the closed unit ball of a real Banach space E. $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies x = y = z. We denote by $extB_E$ the set of extreme points of B_E . $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that f(x) = 1 = ||f|| and f(y) < 1for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. $x \in B_E$ is called a *smooth point* of B_E if there is a unique $f \in E^*$ so that f(x) = 1 = ||f||. A mapping $P : E \to \mathbb{R}$ is a continuous 2-homogeneous polynomial if there exists a continuous bilinear form L on the product $E \times E$ such that P(x) = L(x, x) for every $x \in E$. We denote by $\mathcal{L}(^2E)$ the Banach space of all continuous bilinear forms on E endowed with the norm $||L|| = \sup_{||x||=||y||=1} |L(x,y)|$. $\mathcal{L}_s(^2E)$ denotes the closed subspace of $\mathcal{L}(^2E)$ consisting of all continuous symmetric bilinear forms on E. $\mathcal{P}(^2E)$ denotes the Banach space of all continuous 2-homogeneous polynomials from E into \mathbb{R} endowed with the norm $||P|| = \sup_{||x||=1} ||P(x)|$. For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [7].

In 1998, Choi *et al.* [2,3] characterized the extreme points of the unit ball of $\mathcal{P}(^{2}l_{1}^{2})$ and $\mathcal{P}(^{2}l_{2}^{2})$. In 2007, Kim [11] classified the exposed 2-homogeneous polynomials on $\mathcal{P}(^{2}l_{p}^{2})$ $(1 \leq p \leq \infty)$, where $l_{p}^{2} = \mathbb{R}^{2}$ with the l_{p} -norm. Recently, Kim [13, 15, 19] classify the extreme, exposed, smooth points of the unit ball of $\mathcal{P}(^{2}d_{*}(1,w)^{2})$, where $d_{*}(1,w)^{2} = \mathbb{R}^{2}$ with the octagonal norm $\|(x,y)\|_{d_{*}} = \max\{|x|, |y|, \frac{|x|+|y|}{1+w}\}$. In 2009, Kim [12] classified the extreme, exposed, smooth points of the unit ball of $\mathcal{L}_{s}(^{2}l_{\infty}^{2})$. Recently, Kim [14, 16–18, 21]

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classified the extreme, exposed, smooth points of the unit balls of $\mathcal{L}_s(^2d_*(1,w)^2)$ and $\mathcal{L}(^2d_*(1,w)^2)$.

We refer to ([1–6], [8–26] and references therein) for some recent work about extremal properties of multilinear mappings and homogeneous polynomials on some classical Banach spaces. For 0 < w < 1, $\mathbb{R}^2_{h(w)}$ denotes \mathbb{R}^2 endowed with a hexagonal norm $||(x, y)||_{h(w)} := \max\{|y|, |x| + (1 - w)|y|\}$. In this paper, we classify the extreme bilinear forms of the unit ball of $\mathcal{L}({}^2\mathbb{R}^2_{h(w)})$.

2. Main results

Let 0 < w < 1 and $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}(\mathbb{R}^2_{h(w)})$ for some reals a, b, c, d. For simplicity we will write

$$T((x_1, y_1), (x_2, y_2)) = (a, b, c, d).$$

Let

$$\begin{split} T_1((x_1, y_1), (x_2, y_2)) &:= T((x_2, y_2), (x_1, y_1)) = (a, b, d, c), \\ T_2((x_1, y_1), (x_2, y_2)) &:= T((x_1, -y_1), (x_2, y_2)) = (a, -b, c, -d), \\ T_3((x_1, y_1), (x_2, y_2)) &:= T((x_1, y_1), (x_2, -y_2)) = (a, -b, -c, d), \\ T_4((x_1, y_1), (x_2, y_2)) &:= T((x_1, y_1), (-x_2, -y_2)) = (-a, -b, -c, -d). \end{split}$$

Then $||T_i|| = ||T||$ (i = 1, ..., 4). Hence, without loss of generality, we may assume that $a \ge 0$ and $c \ge d \ge 0$.

Theorem 2.1. Let 0 < w < 1 and $T((x_1, y_1), (x_2, y_2)) := (a, b, c, d) \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ with $a \ge 0$ and $c \ge d \ge 0$. Then

$$||T|| = \max\{a, aw + c, |aw^{2} + b| + (c + d)w, |aw^{2} - b| + (c - d)w\}.$$

Proof. Note that $ext B_{\mathbb{R}^2_{h(w)}} = \{(\pm 1, 0), (w, \pm 1), (-w, \pm 1)\}$. By the Krein-Milman theorem, $B_{\mathbb{R}^2_{h(w)}} = \overline{co}(ext B_{\mathbb{R}^2_{h(w)}})$, where $\overline{co}(A)$ is the closed convex hull of the set A. By the bilinearity of T, it follows that

$$\begin{split} \|T\| &= \max\{|T((\pm 1,0),(\pm 1,0))|,|T((\pm 1,0),(w,\pm 1))|,\\ &|T((w,\pm 1),(\pm 1,0))|,|T((w,\pm 1),(w,\pm 1))|\}\\ &= \max\{|T((1,0),(1,0))|,|T((1,0),(w,1))|,|T((w,1),(1,0))|,\\ &|T((1,0),(w,-1))|,|T((w,-1),(1,0))|,|T((w,1),(w,1))|,\\ &|T((w,-1),(w,-1))|,|T((w,1),(w,-1))|,|T((w,-1),(w,1))|\}\\ &= \max\{|a|,|a|w+|c|,|a|w+|d|,|aw^2+b|+|c+d|w,|aw^2-b|+|c-d|w\}\\ &= \max\{a,aw+c,|aw^2+b|+(c+d)w,|aw^2-b|+(c-d)w\}. \ \Box$$

Note that if ||T|| = 1, then $|a| \le 1, |b| \le 1, |c| \le 1$ and $|d| \le 1$.

Theorem 2.2. Let 0 < w < 1 and $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$. Then the followings are equivalent:

(1) T is extreme;

(2) (a, b, d, c) is extreme;

- (3) (a, -b, c, -d) is extreme;
- (4) (a, -b, -c, d) is extreme;
- (5) (-a, -b, -c, -d) is extreme.

Proof. It follows from Theorem 2.1 and the remark above of Theorem 2.1. \Box

Theorem 2.3 ([20, Theorem 2.3]). Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + c(x_1y_2 + x_2y_1) \in \mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})$ with $a \ge 0, c \ge 0$.

(a) Let $0 < w < \frac{1}{2}$. Then, $T \in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}$ if and only if

$$T \in \{(0, \pm 1, 0, 0), (1, (1 - w)^2, 1 - w, 1 - w), (1, 1 - w^2, 0, 0), (1, w^2 - 1, w, w), (0, 1 - 2w, 1, 1), (1, -3w^2 + 2w - 1, 1 - w, 1 - w)\}.$$

(b) Let $w = \frac{1}{2}$. Then, $T \in ext B_{\mathcal{L}_s(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ if and only if

$$T \in \{(0,\pm 1,0,0), (1,\frac{1}{4},\frac{1}{2},\frac{1}{2}), (1,\frac{3}{4},0,0), (0,0,1,1), (1,-\frac{3}{4},\frac{1}{2},\frac{1}{2})\}.$$

(c) Let $\frac{1}{2} < w < 1$. Then, $T \in extB_{\mathcal{L}_s(^2\mathbb{R}^2_{h(w)})}$ if and only if

$$T \in \{(0, \pm 1, 0, 0), (1, (1 - w)^2, 1 - w, 1 - w), (1, 1 - w^2, 0, 0), \\(1, w^2 - 1, 1 - w, 1 - w), (\frac{1}{2w}, \frac{w - 2}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{2w - 1}{2w^2}, \frac{1 - 2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}$$

Let

$$\begin{aligned} Norm(T) \\ &= \{((x_1, y_1), (x_2, y_2)) \in \{((1, 0), (1, 0)), ((1, 0), (w, 1)), ((w, 1), (1, 0)), \\ &\quad ((1, 0), (w, -1)), ((w, -1), (1, 0)), ((w, 1), (w, 1)), ((w, -1), (w, -1)), \\ &\quad ((w, 1), (w, -1)), ((w, -1), (w, 1))\} : |T((x_1, y_1), (x_2, y_2))| = \|T\| \}. \end{aligned}$$

We call Norm(T) the norming set of T.

Theorem 2.4. Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ with $a \ge 0$ and $c \ge d \ge 0$. (a) Let $0 < w < \frac{1}{2}$. Then, $T \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$ if and only if

$$\begin{split} T &\in \{(0,\pm(1-w),1,0),(1,-(w^2-w+1),1-w,w),(0,\pm1,0,0),\\ &(1,(1-w)^2,1-w,1-w),(1,w^2-1,w,w),(0,1-2w,1,1),\\ &(0,-1+2w,1,1),(1,-3w^2+2w-1,1-w,1-w)\}. \end{split}$$

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(b) Let
$$w = \frac{1}{2}$$
. Then, $T \in ext B_{\mathcal{L}({}^{2}\mathbb{R}^{2}_{h(\frac{1}{2})})}$ if and only if
 $T \in \{(0, \pm \frac{1}{2}, 1, 0), (0, \pm 1, 0, 0), (1, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}), (0, 0, 1, 1), (1, -\frac{3}{4}, \frac{1}{2}, \frac{1}{2})\}.$
(c) Let $\frac{1}{2} < w < 1$. Then, $T \in ext B_{\mathcal{L}({}^{2}\mathbb{R}^{2}_{h(w)})}$ if and only if
 $T \in \{(0, \pm (1 - w), 1, 0), (0, \pm 1, 0, 0), (1, (1 - w)^{2}, 1 - w, 1 - w), (1, w^{2} - 1, 1 - w, 1 - w), (\frac{1}{2w}, \frac{w - 2}{2}, \frac{1}{2}, \frac{1}{2}), (0, 0, 1, \frac{1}{w} - 1), (\frac{2w - 1}{2w^{2}}, \frac{1 - 2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}.$

Proof. Let $T((x_1, y_1), (x_2, y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1 \in \mathcal{L}({}^2\mathbb{R}^2_{h(w)})$ with ||T|| = 1. By the remark above of Theorem 2.1, we may assume that $a \ge 0$ and $c \ge d \ge 0$. Suppose that T is extreme. In what follows we will distinguish three cases: (c > d = 0) or (c > d > 0) or $(c = d \ge 0)$.

 $\underline{\text{Case 1}}: c > d = 0.$

By the remark above of Theorem 2.1, we may assume that $b \ge 0$. Note that $1 = \max\{a, aw + c, |aw^2 + b| + cw, |aw^2 - b| + cw\}$. We have the following subcases to consider:

(a = 0, b = 0) or (a = 0, b > 0) or (a > 0, b = 0) or (a > 0, b > 0).

Subcase 1: a = 0, b = 0.

Then, $1 = \max\{c, cw\}$, hence, T = (0, 0, 1, 0), which is not extreme. Indeed, let $T_1 = (0, \frac{1}{n}, 1, 0), T_2 = (0, -\frac{1}{n}, 1, 0)$ for $n \in \mathbb{N}$ with $\frac{1}{n} + w < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Subcase 2: a = 0, b > 0.

Then, $1 = \max\{c, b + cw\}$, hence, T = (0, 1 - w, 1, 0).

Claim: T = (0, 1 - w, 1, 0) is extreme.

Let $T_1 = (\epsilon, (1 - w) + \delta, 1 + \gamma, \rho), T_2 = (-\epsilon, (1 - w) - \delta, 1 - \gamma, -\rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (w,1))| \le 1, |T_i((1,0), (w,-1))| \le 1, |T_i((w,1), (w,1))| \le 1, |T_i((w,1), (w,-1))| \le 1$, we have

$$w\epsilon + \gamma = 0,$$

$$w\epsilon - \gamma = 0,$$

$$w^{2}\epsilon + \delta + w\gamma + w\rho = 0,$$

$$w^{2}\epsilon - \delta - w\gamma + w\rho = 0.$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$. Hence, T = (0, 1 - w, 1, 0) is extreme. By Theorem 2.2, (0, -(1 - w), 1, 0) is extreme.

Subcase 3: a > 0, b = 0.

Then, $1 = \max\{a, aw + c, aw^2 + cw\}$. Note that $aw^2 + cw \le aw^2 + (1 - aw)w = w$. Let $T_1 = (a, \frac{1}{n}, c, \frac{1}{n}), T_2 = (a, -\frac{1}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with

 $aw^2 + cw + \frac{1+w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Subcase 4: a > 0, b > 0.

Then, $1 = \max\{a, aw + c, aw^2 + b + cw\}$. Note that

 $Norm(T) \subseteq \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (w,1))\}.$

Then, $|aw^2 - b| + cw < 1$. If $aw^2 - b > 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 =$ $\begin{array}{l} (a,b+\frac{w}{n},c,-\frac{1}{n}) \text{ for some } n \in \mathbb{N} \text{ with } 0 < b-\frac{w}{n}, b+\frac{1}{n} < 1, 0 < c-\frac{1}{n}, c+\frac{1}{n} < 1, aw^2+b-\frac{w}{n} > 0, aw^2-b-\frac{w}{n} > 0. \text{ Then } T=\frac{1}{2}(T_1+T_2) \text{ and } \|T_i\|=1 \text{ for } T=\frac{1}{2}(T_1+T_2) \text{ for } T=\frac{1}{2}(T_1+T_$ i = 1, 2, which is a contradiction. If $aw^2 - b = 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 = 0$ $(a, b + \frac{w}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with $cw + \frac{2w}{n} < 1, 0 < b - \frac{w}{n}, b + \frac{1}{n} < 1, 0 < c - \frac{1}{n}, c + \frac{1}{n} < 1, aw^2 + b - \frac{w}{n} > 0$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

If $aw^2 - b < 0$, let $T_1 = (a, b - \frac{w}{n}, c, \frac{1}{n}), T_2 = (a, b + \frac{w}{n}, c, -\frac{1}{n})$ for some $n \in \mathbb{N}$ with $0 < b - \frac{w}{n}, b + \frac{1}{n} < 1, 0 < c - \frac{1}{n}, c + \frac{1}{n} < 1, aw^2 + b - \frac{w}{n} > 0, aw^2 - b + \frac{w}{n} < 0, |aw^2 - b| + cw + \frac{2w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Case 2: c > d > 0.

We have the following subcases to consider:

 $(0 \le a < 1, b \ge 0)$ or $(a = 1, b \ge 0)$ or $(0 \le a < 1, b < 0)$ or (a = 1, b < 0). Subcase 1: $0 \le a < 1, b \ge 0$.

Note that $1 = \max\{aw + c, aw^2 + b + (c + d)w\}$. Calculation shows that if b < 0, then $Norm(T) \subseteq \{((1,0), (w,1)), ((w,1), (w,1)), ((w,-1), (w,1))\}$, and if (a = 0, b > 0) or $(a \neq 0, b \ge 0)$, then $Norm(T) \subseteq \{((1, 0), (w, 1)), ((w, 1), (w, 1)), ((w, 1), (w, 1))\}$ $\begin{array}{l} (a = 0, 0 \neq 0) \text{ or } (a \neq 0, 0 \neq 0), \text{ then } 1 \neq 0 \text{ or } (a \neq 0, 0 \neq 0), \text{ then } 1 \neq 0 \text{ or } (a \neq 1), ((a, 1)), ((a, 1)$ Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Therefore, a = 0 = b, and $1 = \max\{c, (c+d)w\}$. Since T is extreme, c =1 = (c+d)w. Hence, $T = (0, 0, 1, \frac{1}{w} - 1)$.

Claim: $T = (0, 0, 1, \frac{1}{w} - 1)$ is extreme for $\frac{1}{2} < w < 1$ Let $T_1 = (\epsilon, \delta, 1 + \gamma, \frac{1}{w} - 1 + \rho), T_2 = (-\epsilon, -\delta, 1 - \gamma, \frac{1}{w} - 1 - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (w,1))| \leq 1$ $1, |T_i((1,0),(w,-1))| \le 1, |T_i((w,1),(w,1))| \le 1, |T_i((w,-1),(w,-1))| \le 1,$ we have

$$w\epsilon + \gamma = 0,$$

$$w\epsilon - \gamma = 0,$$

$$w^{2}\epsilon + \delta + w\gamma + w\rho = 0,$$

$$w^{2}\epsilon + \delta - w\gamma - w\rho = 0,$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Subcase 2: $a = 1, b \ge 0$.

Note that $1 = \max\{a, aw + c, aw^2 + b + (c + d)w\}$. Hence,

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,1),(w,1))\}$$

Let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $b + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Subcase 3: $0 \le a < 1, b < 0.$

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Note that $1 = \max\{aw + c, |aw^2 + b| + (c + d)w, aw^2 - b + (c - d)w\}$. First, suppose that $aw^2 + b \ge 0$. Then, $1 = \max\{aw + c, aw^2 + b + (c + d)w, aw^2 - b + (c - d)w\}$. Hence,

$$Norm(T) \subseteq \{((1,0), (w,1)), ((w,1), (w,1)), ((w,-1), (w,1))\}.$$

Let $T_1 = (a + \frac{1}{n}, b - \frac{w}{n}, c - \frac{w}{n}, d + \frac{1}{n}), T_2 = (a - \frac{1}{n}, b + \frac{w}{n}, c + \frac{w}{n}, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $a + \frac{1}{n} < 1, b + \frac{w}{n} < 1, c + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction. Next, suppose that $aw^2 + b < 0$. Then, $1 = \max\{aw + c, -(aw^2 + b) + (c + d)w, aw^2 - b + (c - d)w\}$. Hence,

$$Norm(T) \subseteq \{((1,0), (w,1)), ((w,-1), (w,-1)), ((w,-1), (w,1))\}.$$

Let $T_1 = (a + \frac{1}{n}, b - \frac{w^2}{n}, c - \frac{w}{n}, d + \frac{w}{n}), T_2 = (a - \frac{1}{n}, b + \frac{w^2}{n}, c + \frac{w}{n}, d - \frac{w}{n})$ for some $n \in \mathbb{N}$ with $a + \frac{1}{n} < 1, b + \frac{w^2}{n} < 1, c + \frac{w}{n} < 1, d + \frac{w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Subcase 4: a = 1, b < 0.

Note that $1 = \max\{a, w+c, |w^2+b|+(c+d)w, w^2-b+(c-d)w\}$. First, suppose that $w^2+b \ge 0$. Then, $1 = \max\{a, w+c, w^2+b+(c+d)w, w^2-b+(c-d)w\}$. Hence,

 $Norm(T) \subseteq \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (w,1)), ((w,-1), (w,1))\}.$

If $w^2 + b > 0$, let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $w^2 + b - \frac{w}{n} > 0$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction. Suppose that $w^2 + b = 0$. We claim that (c + d)w < 1. If (c + d)w = 1, then $1 = cw + dw < c + w \le 1$, which is impossible. Let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $(c + d)w + \frac{2w}{n} < 1, w^2 - b + \frac{w}{n} > 0, b + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction.

Next, suppose that $w^2 + b < 0$. Then, $1 = \max\{a, w + c, -(w^2 + b) + (c + d)w, w^2 - b + (c - d)w\}$. We will show that

 $Norm(T) = \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,-1)),((w,-1),(w,1))\}.$

Otherwise, we have the following subcases to consider:

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,-1))\}$$

or

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,1))\}$$

or

$$Norm(T) \subseteq \{((1,0), (1,0)), ((w,-1), (w,-1)), ((w,-1), (w,1))\}$$

First, suppose that

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,-1))\}.$$

Let $T_1 = (1, b + \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b - \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $w^2 + b + \frac{w}{n} < 0, w^2 - b - \frac{w}{n} > 0, |b| + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction. Suppose that

$$Norm(T) \subseteq \{((1,0),(1,0)),((1,0),(w,1)),((w,-1),(w,1))\}.$$

Then, $-(w^2 + b) + (c + d)w < 1$. Let $T_1 = (1, b - \frac{w}{n}, c, d + \frac{1}{n}), T_2 = (1, b + \frac{w}{n}, c, d - \frac{1}{n})$ for some $n \in \mathbb{N}$ with $-(w^2 + b) + (c + d)w + \frac{2w}{n} < 1, |b| + \frac{w}{n} < 1, d + \frac{1}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction. Suppose that

$$Norm(T) \subseteq \{((1,0),(1,0)),((w,-1),(w,-1)),((w,-1),(w,1))\}.$$

Then, w + c < 1. Let $T_1 = (1, b + \frac{w}{n}, c + \frac{1}{n}, d), T_2 = (1, b - \frac{w}{n}, c - \frac{1}{n}, d)$ for some $n \in \mathbb{N}$ with $w + c + \frac{1}{n} < 1, w^2 + b + \frac{w}{n} < 0, w^2 - b - \frac{w}{n} > 0, |b| + \frac{w}{n} < 1$. Then $T = \frac{1}{2}(T_1 + T_2)$ and $||T_i|| = 1$ for i = 1, 2, which is a contradiction. Therefore, we conclude the claim. By a calculation, $T = (1, -(w^2 - w + 1), 1 - w, w)$ for $w < \frac{1}{2}$.

Claim: $T = (1, -(1 - w + w^2), 1 - w, w)$ is extreme for $w < \frac{1}{2}$.

Let $T_1 = (1 + \epsilon, -(1 - w + w^2) + \delta, 1 - w + \gamma, w + \rho), T_2 = (1 - \epsilon, -(1 - w + w^2) - \delta, 1 - w - \gamma, w - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (1,0))| \le 1, |T_i((1,0), (w,1))| \le 1, |T_i((w,-1), (w,1))| \le 1, |T_i((w,-1), (w,-1))| \le 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ v^2\epsilon - \delta + w\gamma - w\rho &= 0, \\ v^2\epsilon + \delta - w\gamma - w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

<u>Case 3</u>: $c = d \ge 0$.

Since $T = (a, b, c, c) \in extB_{\mathcal{L}(2\mathbb{R}^2_{h(w)})}$, then $T \in extB_{\mathcal{L}_s(2\mathbb{R}^2_{h(w)})}$. By Theorem 2.3, for $0 < w \leq \frac{1}{2}$,

$$\begin{split} T &\in \{(0,\pm 1,0,0), (1,(1-w)^2,1-w,1-w), (1,1-w^2,0,0), \\ &\quad (1,w^2-1,w,w), (0,1-2w,1,1), (1,-3w^2+2w-1,1-w,1-w)\} \end{split}$$

and for $\frac{1}{2} < w < 1$, $T \in extB_{\mathcal{L}_s({}^2\mathbb{R}^2_{h(w)})}$ if and only if $T \in \{(0, \pm 1, 0, 0), (1, (1-w)^2, 1-w, 1-w), (1, 1-w^2, 0, 0), (1, 1-w^2, 0$

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$$(1, w^2 - 1, 1 - w, 1 - w), (\frac{1}{2w}, \frac{w - 2}{2}, \frac{1}{2}, \frac{1}{2}), (\frac{2w - 1}{2w^2}, \frac{1 - 2w}{2}, \frac{1}{2w}, \frac{1}{2w})\}$$

Claim: $(1, 1 - w^2, 0, 0) \notin B_{\mathcal{L}(2\mathbb{R}^2_{h(w)})}$ for 0 < w < 1.

Let $T_1 = (1, 1 - w^2, -\frac{1}{n}, \frac{1}{n})$ and $T_2 = (1, 1 - w^2, \frac{1}{n}, -\frac{1}{n})$ for a sufficiently large $n \in \mathbb{N}$ such that $||T_i|| = 1$ for i = 1, 2. Since $(1, 1 - w^2, 0, 0) = \frac{1}{2}(T_1 + T_2)$, $(1, 1 - w^2, 0, 0)$ is not extreme.

Claim: $T = (0, \pm 1, 0, 0) \in ext B_{\mathcal{L}({}^{2}\mathbb{R}^{2}_{h(w)})}$ for 0 < w < 1. Note that

$$Norm(T) = \{((w, 1), (w, 1)), ((w, -1), (w, 1)), ((w, 1), (w, -1)), ((w, -1), (w, -1))\}.$$

Let $T_1 = (\epsilon, 1+\delta, \gamma, \rho)$ and $T_2 = (\epsilon, 1-\delta, -\gamma, -\rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((w, 1), (w, 1))| \le 1$, $|T_i((w, -1), (w, 1))| \le 1$, $|T_i((w, 1), (w, -1))| \le 1, |T_i((w, -1), (w, -1))| \le 1$, we have

$$\begin{split} &w^2\epsilon+\delta+w\gamma+w\rho=0,\\ &w^2\epsilon-\delta+w\gamma-w\rho=0,\\ &w^2\epsilon+\delta-w\gamma-w\rho=0,\\ &w^2\epsilon-\delta-w\gamma+w\rho=0, \end{split}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

First, suppose that $0 < w \leq \frac{1}{2}$. Claim: $T = (1, (1-w)^2, 1-w, 1-w) \in extB_{\mathcal{L}(2\mathbb{R}^2_{h(w)})}$. Note that

$$Norm(T) = \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (1,0)), ((w,1), (w,1))\}$$

Let $T_1 = (1 + \epsilon, (1 - w)^2 + \delta, 1 - w + \gamma, 1 - w + \rho)$ and $T_2 = (1 - \epsilon, (1 - w)^2 - \epsilon)^2$ $\delta, 1 - w - \gamma, 1 - w - \rho$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in$ \mathbb{R} . Since $|T_i((1,0),(1,0))| \le 1$, $|T_i((1,0),(w,1))| \le 1$, $|T_i((w,1),(1,0))| \le 1$, $|T_i((w, 1), (w, 1))| \le 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \end{aligned}$$
$$w^2\epsilon + \delta + w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim:
$$T = (1, w^2 - 1, w, w) \in ext B_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}.$$

Note that

$$Norm(T) = \{((1,0), (1,0)), ((w,-1), (w,-1)), ((w,1), (w,-1)), ((w,-1), (w,1))\}.$$

Let $T_1 = (1+\epsilon, w^2-1+\delta, w+\gamma, w+\rho)$ and $T_2 = (1-\epsilon, w^2-1-\delta, w-\gamma, w-\rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (1,0))| \le 1$, $|T_i((w,-1), (w,-1))| \le 1$, $|T_i((w,1), (w,-1))| \le 1$, $|T_i((w,-1), (w,1))| \le 1$, we have

$$\begin{split} \epsilon &= 0, \\ w^2 \epsilon + \delta - w\gamma - w\rho &= 0, \\ w^2 \epsilon - \delta - w\gamma + w\rho &= 0, \\ w^2 \epsilon - \delta + w\gamma - w\rho &= 0, \end{split}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (0, 1 - 2w, 1, 1) \in ext B_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}.$ Note that

$$Norm(T) = \{((1,0), (w,1)), ((w,1), (1,0)), ((1,0), (w,-1)), ((w,-1), (1,0)), ((w,1), (w,1))\}.$$

Let $T_1 = (\epsilon, 1 - 2w + \delta, 1 + \gamma, 1 + \rho)$ and $T_2 = (-\epsilon, 1 - 2w - \delta, 1 - \gamma, 1 - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (w,1))| \le 1$, $|T_i((w,1), (1,0))| \le 1$, $|T_i((1,0), (w,-1))| \le 1$, $|T_i((w,1), (w,1))| \le 1$, we have

$$w\epsilon + \gamma = 0,$$

$$w\epsilon + \rho = 0,$$

$$w\epsilon - \gamma = 0,$$

$$w^{2}\epsilon + \delta + w\gamma + w\rho = 0,$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (1, -3w^2 + 2w - 1, 1 - w, 1 - w) \in ext B_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}.$ Note that

$$Norm(T) = \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (1,0)), ((w,-1), (w,-1))\}.$$

Let $T_1 = (1+\epsilon, -3w^2+2w-1+\delta, 1-w+\gamma, 1-w+\rho)$ and $T_2 = (1-\epsilon, -3w^2+2w-1-\delta, 1-w-\gamma, 1-w-\rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (1,0))| \le 1$, $|T_i((1,0), (w,1))| \le 1$, $|T_i((w,1), (1,0))| \le 1$, $|T_i((w,-1), (w,-1))| \le 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon + \delta - w\gamma - w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Next, suppose that $\frac{1}{2} < w < 1$. Claim: $T = (1, (1-w)^2, 1-w, 1-w) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}$. Note that

 $Norm(T) = \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (1,0)), ((w,1), (w,1))\}.$ Let $T_1 = (1 + \epsilon, (1 - w)^2 + \delta, 1 - w + \gamma, 1 - w + \rho)$ and $T_2 = (1 - \epsilon, (1 - w)^2 - \delta, 1 - w - \gamma, 1 - w - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (1,0))| \le 1$, $|T_i((1,0), (w,1))| \le 1$, $|T_i((w,1), (1,0))| \le 1$, $|T_i((w,1), (w,1))| \le 1$, we have

$$\begin{split} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon + \delta + w\gamma + w\rho &= 0, \end{split}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (1, w^2 - 1, 1 - w, 1 - w) \in ext B_{\mathcal{L}(^2\mathbb{R}^2_{h(w)})}$. Note that

$$Norm(T) = \{((1,0), (1,0)), ((1,0), (w,1)), ((w,1), (1,0)), ((w,1), (w,-1)), ((w,-1), (w,1))\}.$$

Let $T_1 = (1 + \epsilon, w^2 - 1 + \delta, 1 - w + \gamma, 1 - w + \rho)$ and $T_2 = (1 - \epsilon, w^2 - 1 - \delta, 1 - w - \gamma, 1 - w - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (1,0))| \le 1$, $|T_i((1,0), (w,1))| \le 1$, $|T_i((w,1), (1,0))| \le 1$, $|T_i((w,1), (w,-1))| \le 1$, we have

$$\begin{aligned} \epsilon &= 0, \\ w\epsilon + \gamma &= 0, \\ w\epsilon + \rho &= 0, \\ w^2\epsilon - \delta - w\gamma + w\rho &= 0, \end{aligned}$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim:
$$T = (\frac{1}{2w}, \frac{2-w}{2}, \frac{1}{2}, \frac{1}{2}) \in extB_{\mathcal{L}(2\mathbb{R}^2_{h(w)})}.$$

Note that

$$\begin{split} Norm(T) &= \{ ((1,0), (w,1)), ((w,1), (1,0)), ((w,-1), (w,-1)), ((w,1), (w,-1)) \}.\\ \text{Let } T_1 &= (\frac{1}{2w} + \epsilon, \frac{2-w}{2} + \delta, \frac{1}{2} + \gamma, \frac{1}{2} + \rho) \text{ and } T_2 &= (\frac{1}{2w} - \epsilon, \frac{2-w}{2} - \delta, \frac{1}{2} - \gamma, \frac{1}{2} - \rho) \text{ be such that } \|T_1\| &= 1 = \|T_2\| \text{ for some } \epsilon, \delta, \gamma, \rho \in \mathbb{R}. \text{ Since } |T_i((1,0), (w,1))| \leq 1, \\ |T_i((w,1), (1,0))| &\leq 1, \ |T_i((w,-1), (w,-1))| \leq 1, \ |T_i((w,1), (w,-1))| \leq 1, \text{ we have} \end{split}$$

$$w\epsilon + \gamma = 0,$$

$$w\epsilon + \rho = 0,$$

$$w^{2}\epsilon + \delta - w\gamma - w\rho = 0,$$

$$w^{2}\epsilon - \delta - w\gamma + w\rho = 0,$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$.

Claim: $T = (\frac{2w-1}{2w^2}, \frac{1-2w}{2}, \frac{1}{2w}, \frac{1}{2w}) \in extB_{\mathcal{L}({}^2\mathbb{R}^2_{h(w)})}.$ Note that

 $Norm(T) = \{((1,0),(w,1)),((w,1),(1,0)),((w,1),(w,1)),((w,-1),(w,-1))\}.$ Let $T_1 = (\frac{2w-1}{2w^2} + \epsilon, \frac{1-2w}{2} + \delta, \frac{1}{2w} + \gamma, \frac{1}{2w} + \rho)$ and $T_2 = (\frac{2w-1}{2w^2} - \epsilon, \frac{1-2w}{2} - \delta, \frac{1}{2w} - \gamma, \frac{1}{2w} - \rho)$ be such that $||T_1|| = 1 = ||T_2||$ for some $\epsilon, \delta, \gamma, \rho \in \mathbb{R}$. Since $|T_i((1,0), (w,1))| \le 1$, $|T_i((w,1), (1,0))| \le 1$, $|T_i((w,1), (w,1))| \le 1$, $|T_i((w, -1), (w, -1))| \le 1$, we have

$$w\epsilon + \gamma = 0,$$

$$w\epsilon + \rho = 0,$$

$$w^{2}\epsilon + \delta + w\gamma + w\rho = 0,$$

$$w^{2}\epsilon + \delta - w\gamma - w\rho = 0,$$

which shows that $0 = \epsilon = \delta = \gamma = \rho$. Therefore, we complete the proof.

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